Thinking Classroom – How can teachers enhance students' engagement and joy of learning through creative thinking, analytical thinking, reflective thinking and problem-solving?

For Students at St Andrew's Junior School, Singapore

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Contents

Biography

Peng Seng is the Year Head of St Andrew's Junior School from Singapore with more than 14 years of experience in the classroom. He oversees the holistic development of his students at the Primary 3 and 4 levels. His awards include

- 2015 Ministry of Education (MOE) – Sponsored Training
- 2018 MOE (Schools) Award – Let me show you how I solved it!
- 2019 National Day Award - The Commendation Medal

He believes in equipping all his students with 21st-century skills to prepare them for the ever-changing future. In addition to his primary job functions, he runs numerous workshops for teachers in his schools and MOE.

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Executive Summary

This inquiry examines how the Singaporean mathematics curriculum should focus on problem-solving, reasoning and proof, communication, connection, and representations. Accomplishing this goal would involve moving mathematics instruction away from drill and practice toward teaching mathematics through problem-solving. It also involves moving mathematics instruction away from procedural fluency to conceptual understanding. The comments on this paper are based on my classroom experience, my learning thus far in the United States and the experiences of my partner teachers. First, I will examine why mathematics teaching needs to emphasise developing thinking skills. Second, I will offer suggestions to help teachers implement the revised mathematics curriculum focused on teaching through problem-solving and cultivating the joy of learning mathematics.

Volatility, uncertainty, complexity, and ambiguity (VUCA) describes the situation of constant, unpredictable change that is now the norm in which organisations and institutions function today. Students must continue to stay flexible and agile in order to navigate this VUCA world. To help our students thrive in this fast-changing world, the Ministry of Education Singapore has identified a suite of core values and competencies that are increasingly important so schools can better our students for the future. As a result, the teaching method in class must reflect the balance between mathematics knowledge and 21st-century skills.

The joy of learning is an intrinsic motivation that pushes students to explore, which leads to discovering their interests and passions (Ministry of Education, 2017). During the Committee Supply Debate in Parliament in March 2017, Singapore's former Education Minister (Schools) Ng Chee Meng suggested nurturing the joy of learning so that students can be intrinsically motivated. It suggests that nurturing the joy of learning would be a priority for Singapore's education system moving forward. In a classroom where the joy of learning exists, students are challenged and given the autonomy to make decisions about their learning. When students experience the joy of learning, students are engaged in a positive learning experience. All students have the right to access positive learning experiences where they will be engaged in rich mathematics tasks. Therefore, mathematical instruction should foster the use of thinking skills and meaningful learning from every student.

Overview and Relevance of the Topic

Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000) state that "problem-solving is central to inquiry and application and should be interwoven throughout the mathematics curriculum to provide a context for learning and applying mathematical ideas" (p. 256). My inquiry addresses the challenge of teachers knowing how to integrate problem-solving meaningfully into the mathematics curriculum. This inquiry is critical because students must be taught strategies to solve complex, not well-defined problems and lack a clear solution and approach.

Background

The world we live in and work in has changed and will continue to change dramatically, as seen in the recent Covid-19 pandemic. We currently know that, while the public school education system has been around for more than 150 years, the basic schooling model remains the same as a generation or even two generations ago. Students are still taught in a standardised way. In fact, the overall educational system has changed in many ways, but how students are taught has not.

Globalisation, changing demographics, and technological advancements are some of the fundamental driving forces of our current times and will continue to shape our future. Our students must be prepared to face these challenges and seize new and exciting opportunities. Skills needed in the 21st-century workplace are less about being able to compute and more about being able to design solution strategies. Priorities for students in a globalised world include civic literacy, global awareness and cross-cultural skills, critical and inventive thinking, and communication, collaboration and information skills.

According to the National Council of Teachers of Mathematics (NCTM) (2010), the term “problem solving” refers to mathematical tasks that have the potential to provide intellectual challenges for enhancing students’ mathematical understanding and development. Nevertheless, too often mathematics teaching still follows the pattern of the teacher showing one way to do a skill and students practising that skill using the same procedure. Moreover, this approach does not prepare students for their 21st-century lives.

There are three ways to solve problems: teaching for problem-solving, teaching about problem-solving and teaching through problem-solving. Teaching *for* problem-solving begins with learning a skill. For example, students learn to multiply a three-digit number by a one-digit number, and the word problem they solve is a multiplication problem. The major shortcoming of this approach is that students learn very early in school that the problems they encounter will be solved using the skill they just learned. Therefore, there is no point in reading the word problem to see what is happening and what needs to be solved. The numbers simply can be lifted, and the skill applied.

Teaching *about* problem-solving begins with suggested strategies to solve a problem. For example, “draw a picture,” “make a table,” etc. George Polya, a famous mathematician, outlined four steps for problem-solving (1945). They are 1) read the problem, 2) devise a plan, 3) solve the problem, and 4) check work. While there may be evidence that explicit teaching of these four steps can improve students’ abilities to think mathematically, students may see a word problem as a separate endeavour and focus on the steps to follow rather than the mathematics.

Teaching *through* problem-solving focuses students’ attention on ideas and sense-making and develops mathematical practices. Teaching through problem solving also develops a student’s confidence and builds on their strengths. It allows for collaboration among students and engages students in their learning. This means that students learn mathematics through inquiry, where they explore real contexts, problems, situations, and models and, from those explorations, they learn mathematics. When teaching *through* problem-solving, students are focused on ideas and sense-making while developing confidence in mathematics.

Literature Review

Students’ Creative Thinking in Classroom Mathematics

Regarding creative thinking in classroom mathematics, Tatag (2011) suggests that people exhibit different degrees of creative thinking and are creative despite their various backgrounds and other abilities. Students, therefore, have different levels of creative thinking. Critical thinking examines, relates, and evaluates all aspects of a situation; it goes beyond memorisation. On the other hand, creative thinking is original and reflective thinking. It involves students generating new ideas and concepts for a complex problem. Tatag’s research (2011) gives insight into the characteristics of students’ creative thinking levels and how they might look in a mathematics classroom. Nevertheless, creativity is far from a straightforward construct within educational psychology. Tatag (2011) concludes that identifying, assessing, or classifying students’ creative thinking has many limitations. His research contributes an essential view of how creative thinking can be fostered and measured in the classroom.

Relationship Between Reflective Thinking Skills Towards Problem-Solving and Attitudes Towards Mathematics

Demirel et al. (2015) examined the difference between male and female students' reflective thinking skills towards problem-solving and their attitudes towards mathematics. While there is no significant difference between students' reflective thinking skills towards problem-solving and their gender, there is a significant difference in favour of male students regarding their attitudes towards learning mathematics. This observation has a significant implication for teachers like me teaching in an all-boys school. According to Taggart and Wilson (1999), reflective thinking is the process of making decisions on educational matters and then reflecting upon the decisions made. Students who can think can also solve problems effectively. Demirel et al. (2015) concluded that developing reflective thinking skills is not enough, and teachers need more effort to develop students' reflective thinking skills towards problem-solving. They also suggest increasing the number of curriculum hours for students to use reflective thinking skills to solve real-life problems. This article contributes insights into how to develop problem solvers, especially for male students and teachers like me teaching at an all-boys school.

The Advantage of Analytic Thinkers When Learning Mathematics

Classroom teachers, though their pedagogical approaches, can affect how students learn mathematics (Huincahue et al., 2021). In their study of the relationship between teaching practice and deliberate teaching of thinking skills, the authors found a positive correlation between analytical thinking style and grades. They further concluded that developing analytical thinking skills in schools can translate to academic success because analytical skills are in high demand among employers and are valued in school systems' assessment processes (Huincahue et al., 2021). However, they noted that confidence in self-efficacy does not necessarily mean that high levels of achievement or grades could be obtained. In other words, students with high grades would show high levels of confidence in self-efficacy. Still, high confidence levels in self-efficacy do not equate to academic success.

Higher-Order Thinking Skills

Apino and Retnawati's (2017) study showed that instructional design that includes involving students in solving non-routine problems, developing analytical abilities (analytical thinking), evaluating learning (reflective thinking), creating (creative thinking), and encouraging knowledge construction is effective in improving higher-order thinking skills (HOTS) in mathematics classes. The Singapore Mathematics Curriculum is a spiral program where concepts and skills are revisited and built upon at each level to achieve greater depth and understanding. Thinking skills, such as classifying, comparing, analysing parts and whole, identifying patterns and relationships, induction, deduction, generalising, and spatial visualisation are used in throughout the curriculum (Ministry of Education Singapore, 2013). While the types of thinking skills taught are not clearly defined, the thinking skills can easily be recognised by their characteristics. The presence of meaningful learning activities where students actively engage in deep discussion to construct knowledge and solutions also increases students' higher-order thinking skills. Students become more involved in the learning process because they feel challenged by a given problem, thus leading to increased motivation to learn. By applying the instructional design principle detailed by Apino and Retnawati (2017), learning becomes meaningful for students.

Teaching Through Problem-Solving

Teaching through problem-solving is a pedagogy that leverages problem-solving as a tool to make mathematics learning and practices meaningful (Fi & Degner, 2012). It is not just an effective way to teach mathematics but also provides students with a way to learn mathematics with understanding. Through problem-solving, it is possible to achieve breadth of knowledge within the intended curriculum and depth of understanding; however, it should be viewed not as an add-on activity but as a means of delivering important content. Doing so can give every student in every teacher's classroom the opportunity to be engaged and actively involved in doing mathematics instead of watching the teacher doing all the work.

Invented Strategies Can Develop Meaningful Mathematical Procedures

Conceptual understanding and procedural fluency are widely recognised as essential to teaching and learning mathematics. Conceptual understanding helps students organise their ideas and knowledge into a logical sequence, and procedural fluency allows students to find the right solution to a problem. As noted in *Principles to Actions*, "Conceptual understanding establishes the foundation, and is necessary for developing procedural fluency." (NCTM, 2015, p. 42) While it is important for students to get the correct answers, it is more important for them to understand solutions and explain how and why the solution works. Carroll & Porter (1997) suggested that one way for students to improve their understanding of numbers and operations is to encourage students to develop meaningful computational procedures rather than one standardised procedure. My observation in the classroom as part of my field experience showed that students can develop their solution procedures for multi-digit operations when given the opportunity. Additionally, I noted that when more emphasis is placed on exploring different strategies than on memorising a standard algorithm, students are more engaged. *Principles to Actions* explores other activities and strategies that could promote students' conceptual understanding, discourages educators from teaching a standard algorithm, and encourages teachers to allow students to choose their strategies.

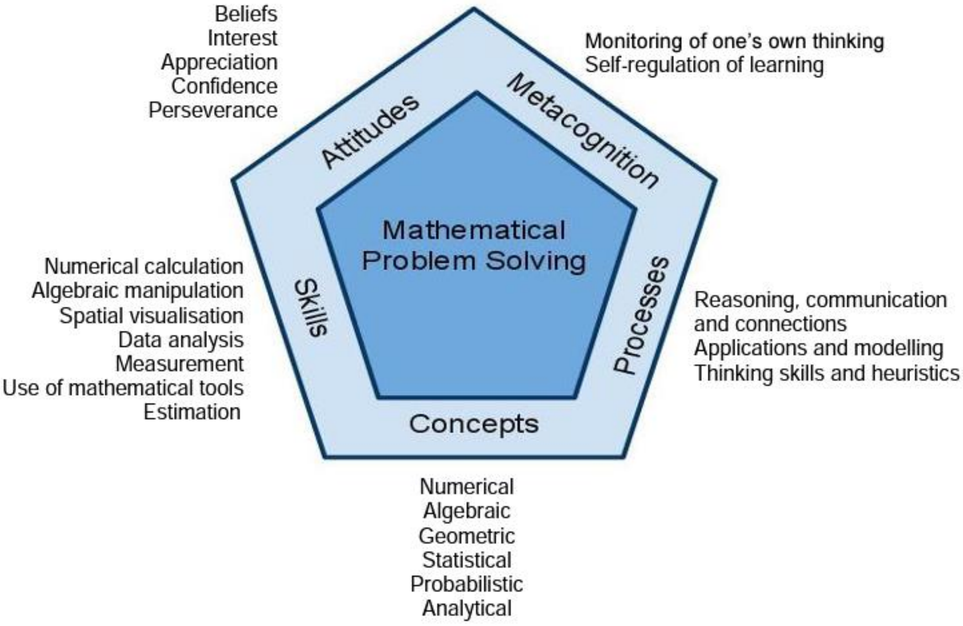
Understanding Singapore's Mathematics Syllabus and Framework

A mathematics framework with five interrelated components and problem-solving as its central focus has been a central focus of Singapore's Mathematics Framework since 1990 (Figure 1). The framework, which is still relevant today, focuses on conceptual understanding, skills proficiency, mathematical processes, attitudes, and metacognition. The content is presented as skills and processes.

To develop mathematical proficiency, students should have the opportunity to use and practice skills, which should not be taught as procedural steps but rather for conceptual understanding.

Figure 1

Mathematics Framework from Singapore's Mathematics Curriculum (Ministry of Education, Singapore, 2013, p. 14)



While problem-solving has been the curriculum's focus since 1990, teachers' implementation of problem-solving is limited by time constraints and uneven educator performance in the classroom. Over the years, 15% of the Primary School Leaving Examinations (PSLE) has consisted of challenging questions (Today, 2022). Teachers are now challenged to design questions that cater to students of different abilities and allow them to think critically and logically to solve problems. In shaping students' attitudes towards mathematics, the Ministry of Education (2013) in Singapore has explicitly spelt out the importance of learning experiences in shaping students' attitudes towards mathematics and the processes and skills required for learning each topic.

Developing Strategies for Multiplication and Division Computation

Multi-digit addition and subtraction are introduced in grades 1 to 3 in Singapore, whereas multi-digit multiplication and division are introduced in grades 3 to 5. Students have difficulty, however, following the steps in the standard algorithms that they do not fully understand. These algorithms have been a staple for years in teaching number operations; nevertheless, these algorithms are rarely used after students leave school. Students who are allowed to compute multi-digit multiplication and division in various ways develop as independent learners (Van de Walle et al., 2019). Executing computations includes getting the students to keep written records of their work, explain their thinking, and discuss the benefits of one strategy over another. Executing computations does not mean just going through the motions of a step-by-step algorithm. While it initially may take a considerable amount of time for students to use invented strategies for multiplication and division problems, at some point students will begin to understand these algorithms' concepts. Using invented strategies not only develops strong number sense in students as they think about multi-digit problems conceptually. Invented strategies also help students develop a long-term understanding of the concepts of multi-digit problems and not fall back on a meaningless step-by-step procedure. The strategies can be found in Annexes A and B.

Three-Phase Lesson Format

Van de Walle et al. (2019) advocated for a three-phase lesson format which provides structure for lessons where students focus on a topic of inquiry, engage in action, and follow up with discussion, reflection and connections. They refer to these three phases as before, during and after. The three phases include the following stages:

1. Getting ready (before) – Activate prior knowledge, be sure the problem is understood, and establish clear expectations.

2. Students work (during) – Let go! Notice students' mathematical thinking, offer appropriate support and provide worthwhile extensions.

3. Class discussion (after) - Promote a mathematical community of learners, listen actively without evaluation, summarise main ideas, and identify future problems. (p.56)

Math In 3-Acts

Materials And Methods

The method by which teachers can integrate the above approaches and content in a mathematics classroom is to undertake a series of Three Act Tasks with their students. The Three-Act Task (Meyer, 2009) was originally used in secondary classrooms to promote thinking skills and is an exciting problem-solving structure specifically designed to engage students in mathematical modelling. This whole-group mathematics activity consists of three distinct parts: an engaging and perplexing Act One, an information and solution-seeking Act Two, and a solution discussion and solution-revealing Act Three. Each act in a Three-Act Task sparks different levels of curiosity and engagement in class.

Act One

This opening act begins with the teacher creating anticipation with a visual aid, which opens up an inquiry by asking the students what they notice and wonder. In this act, students are encouraged to make low and high estimates in response to the teacher's inquiry about the phenomenon that is being observed. The discourse that takes place is essential, and even shy or reluctant students have the confidence and courage to share their thoughts and ask questions. All students' voices matter during a Three-Act Task, and all students' voices are welcomed to contribute to sense-making during Act One.

Act Two

In Act Two, students may ask for more information to solve the problem, and teachers give that information. Using this information, students unpack the problem and select an appropriate strategy to solve the task. Students may continue to ask for information that is missing, and teachers should provide that information.

Act Three

During Act Three, the teacher reveals the solution, and the discussion that follows might follow any number of directions. Students might discuss their plans and solutions as well as assumptions they had made. The teacher may compare and connect students' ideas or link these ideas to the core mathematical functions that students might have called upon. Table 1 below lists the teachers' and students' roles during a Three-Act Task.

After the Three-Act Task

After Act Three, students might extend their learning by generating a list of questions they could investigate next or create other tasks based on the original Task. With this approach, students would have the opportunity to engage in creative mathematical thinking. This extension activity could also allow the teacher to reintroduce student questions which were not addressed earlier.

Benefits of the Three-Act Task

This approach not only supports mathematical thinking but also fully addresses the demands of modelling with mathematics as described in the fourth of the Common Core's eight Standards for Mathematical Practice (SMP). Modelling with mathematics involves students identifying real-world mathematical problems, gathering information and determining which information would help them solve that problem. For elementary students, mathematical models might include drawings, bar models, equations and using manipulatives to represent the information and its relationship in a given problem. This structure is similar to the three-phase lesson format discussed above, which provides lessons where students focus on a topic of inquiry, engage in action, and follow up with discussion, reflection and connections (Van de Walle et al., 2019). A sample unit plan that incorporates the three-act task can be found in Annex D.

Table 1

Three-Act Task Student and Teacher Roles

Act One Analyse	Act Two Create	Act Three Reflect
The Students Use the basic "I Notice, I Wonder" brainstorm (NCTM) to analyse and discuss the video or photographs, including possible mathematical features of the situation presented in the problem Decide on a mathematical question to answer about the situation	**The Students** Create workable solutions to answer the question. Think about an alternative answer to the question.	**The Students** Discuss their strategies and solutions. Reflect on their ideas and consider why their solution was the same or different from the answer.
The Teacher Shares a captivating multimedia illustration of a problem through a video or photographs Guides the students towards a single mathematical question that they can all investigate	**The Teacher** Provides more information or resources that students might think about as they work on the focal question	**The Teacher** May compare and connect students' ideas or reveal the answer

Results

The result of this line of inquiry is a realization that Three-Act Tasks hold value for Singaporean math students. In my experimentation with Three-Act Tasks, I noticed three important distinctions between the proposed activity structure in relation to other problem-solving routines. These features suggest why Three-Act Tasks would be a valuable routine for Singaporean elementary school students:

1. Three-Act Tasks make mathematical operations concrete and facilitate real-world connections. When engaging with Three-Act Tasks, students at every level enter into a deep interaction with a problem. In particular, because the tasks ask students to leverage their prior knowledge of the world around them, learners can see how math relates to their lived experiences. Another avenue by which students make connections is through the use of images. For examples, Act 1 uses videos or photographs to present the situation in the problem, providing a different set of entry points for learners at every level than what a traditional word problem offers.
2. Students' ideas are central to each act. The way that Three-Act Tasks are laid out is extremely engaging, and student ideas naturally flow due to a combination of the engaging images and the problem presented. In addition, students naturally will take multiple pathways to get to the solution and develop a growth mindset when seeing multiple pathways to success.
3. The Three-Act Task lesson structure is designed to engage students in Modelling with Mathematics (SMP 4). In Act One, students make sense of the problem and explain what the problem is asking (SMP 1). In Act Two, students consider a variety of tools, choose the most appropriate tool, and use the tool appropriately (e.g., drawing, bar models, manipulatives) to support their problem-solving. In Act Three, students reflect on the reasonableness of their answer based on the context of the problem (SMP 4), explain their thinking, and justify conclusions in ways that are understandable to teachers and peers (SMP 3).

Inquiry Project Implementation

I plan to use the Lesson Study approach as professional development to help elementary mathematics teachers improve and incorporate the new ideas developed from this inquiry into their teaching. The Lesson Study approach is a method of professional development that encourages teachers to reflect on their teaching practice through a cyclical process of collaborative lesson planning, observation, and examination of student learning. Table 2 details some suggested agendas and times to carry out Lesson Study:

Table 2

Agendas and Times for Three-Act Task Lesson Study

Session	**Suggested Agenda**	**Remarks**
Lesson Study Session 1	Plan • Form a team. • Identify goals. • Choose areas of concern • Use data.	• Discuss and decide on lesson study goals/targets in alignment with the theme and focus • Establish areas of concern about the theme and use of data
Lesson Study Session 2	Data collection	• Identify data sources • Detail parameters for observations (e.g., student behaviour checklist, behavioural records) • Plan interviews (e.g., student surveys, focus group). • Identify useful learning artifacts (e.g., test results, alternative assessments, etc.).
Lesson Study Session 3	Analysis of data collected	• Identify root cause for areas of concern– using data (students' work, focused-group discussion with students, staff & parents) • Sourcing for literature review
Lesson Study Session 4	Share and identify appropriate strategies from the literature review	• Members to read up on literature review articles before the next session.
Lesson Study Session 5	Cycle 1: Planning the research lesson & pre-test (part 1).	
Lesson Study Session 6	Cycle 1: Planning the research lesson & pre-test (part 2).	• Finalise all plans. • Administer pre-test.
Lesson Study Session 7	Cycle 1: Teach the research lesson & carry out Lesson Observation Administer post-test 1	• Debrief for the lesson to be done within three days after lesson observation
Lesson Study Session 8	Post Lesson Observation discussion Revise the research lesson	• Share data collected, discuss areas for improvement and

Session	Suggested Agenda	Remarks
		misconceptions and explore practical strategies to improve lesson • Debrief Lesson 1 • Review the lesson plan for cycle 2
Lesson Study Session 9	Cycle 2: Planning the research lesson	
Lesson Study Session 10	Finalise all plans for Lesson Study Cycle 2	
Lesson Study Session 11	Cycle 2: Teach the research lesson & carry out lesson observations	
Lesson Study Session 12	• Post-lesson observation discussion • Review the research lesson	• Continue to share data collected, discuss areas for improvements and misconceptions and explore practical strategies to finalise the most effective strategies to address the identified areas of concern • Future plans
Lesson Study Session 13	• Report writing • Follow up to finalise the report, etc.	• Write the report
Lesson Study Session 14	Finalise report and prepare for sharing	• Prepare slides for sharing
Staff Learning Forum	Present findings at Staff Learning Forum	

The inquiry project will also be presented as part of the Fulbright Inquiry Project Presentation on November 1, 2022; the Teachers' Conference and ExCEL Fest 2023 (TCEF2023) from May 30 to June 1, 2023; as part of Teacher-Led Workshops (TLWs) in August and September of 2023; and at the Form Network Learning communities (NLCs) in October 2023.

Discussion

When it comes to problem solving, slower-progressing students may have more difficulties than middle or high-progressing students, in part because of the need for understanding the problem. In order to think, students need to be taught how to think. To accomplish this end, both teachers and students need to relearn their roles in the classroom. While many teachers may continue to teach through drill and practice, this approach will not yield the desired results; therefore, teaching through a problem-solving approach must be intentionally embedded in the curriculum.

Teachers need time to relearn pedagogical approaches and try new ways of exploring how students think about the concepts being taught. Additionally, teachers need time to address their own biases as educators. Teachers have the ability to impact the engagement of students directly but need to learn how to present the information being taught in a way that students can learn with deep understanding. When students are engaged and experience the 'joy of learning', they become better learners. Instead of asking, "what is the most efficient way to teach mathematics?" teachers should ask, "what is the most efficient way for students to experience success and learning in the mathematics classroom?". Let us focus on the quality of instruction rather than quantity.

Based on my experience as a workshop presenter, the best way to implement change in the classroom is to watch a teacher who can implement problem-solving in that classroom. Unfortunately, participating in a traditional one-shot workshop will not result in the desired change. Instead, teachers should form groups and work collaboratively to examine and reflect on their practice as it relates to pedagogy that supports problem-solving approaches. By engaging in purposeful and sustained developmental activities together, teachers can learn from one another, with one another, and on behalf of others (Jackson & Temperley, 2007). While studies have shown that reduced class size improves student engagement and achievement in several ways, Singapore's former Education Minister (Schools) Ng Chee Meng argued that class sizes are not indicative of learning support and attention students receive (Today, 2018). In fact, teacher quality matters more to student achievement than any other aspect of schooling. Hence, providing high-quality professional development workshops for teachers may be the most important thing schools can do to improve student learning and bring about better teaching.

Conclusion

Fostering joy in learning is a complex process. It involves presenting students with thought-provoking problems and rich mathematical tasks where students are provided with opportunities to engage in exploration, reasoning, and communication. Moving forward, an important step in fostering rich mathematics experiences for every student will be implementing mathematics tasks with high cognitive demand and multiple entry points (i.e., low-floor and high-ceiling tasks). Teachers must make intentional choices about tasks and their implementation and encourage analytical, reflective, and creative thinking that leads to a solution. Finally, I hope this inquiry project will stimulate other educators to continue this inquiry and apply it in the classroom.

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Annexes

ANNEX A – INVENTED STRATEGIES FOR MULTIPLICATION

Invented strategies are strategies invented by children. They are not standard algorithms taught in the classroom or from the textbook. To learn to use invented strategies, students must be allowed to invent their strategies.

Strategy	Example
Complete number strategies **(Doubling or halving)** Doubling capitalises on the distributive property and the associative property.	63 x 5 = 126 + 126 + 63 = 315 63 + 63 = 126 63 + 63 = 126
Partitioning strategies **(5 is half of 10)** Students decompose numbers in various ways that reflect an understanding of place value. Students use known facts of 1, 10 or 100, then half it to find 5 or 50.	18 x 25 = 250 + 125 + 50 + 25 = 450 10 x 25 = 250 5 x 25 = 125 2 x 25 = 50 1 x 25 = 25
Compensation strategies **(Over or under)** Students look for ways to manipulate numbers so that the calculations are easy.	18 x 25 = (20 – 2) x 25 = (20 x 25) – (2 x 25) = 500 – 50 = 450
Partial products Students multiply each digit of a number with each digit of another, where each digit maintains its place value.	18 x 25 = (10 + 8) x (20 + 5) = (10 x 20) + (10 x 5) + (8 x 20) + (8 x 5) = 200 + 50 + 160 + 40 = 450
Partial products **(Keep one)** Rather than break both numbers into their place value, students keep on factor whole.	18 x 25 = (10 + 8) x 25 = (10 x 25) + (8 x 25) = 250 + 200 = 450
Partial products **(Flexible)** Students decompose each factor multiplicatively into smaller factors.	18 x 25 = (9 x 2) x (5 x 5) = 9 x (2 x 5) x 5 = 9 x 10 x 5 = 9 x 50 = 450

ANNEX B – INVENTED STRATEGIES FOR DIVISION

Invented strategies are strategies invented by children. They are not standard algorithms taught in the classroom or from the textbook. To learn to use invented strategies, students must be allowed to invent their strategies.

<table>
<tr><th>Strategy</th><th>Example</th></tr>
<tr><td>Multiplying up / doubling
Students may think of division as multiplication and make use of doubling the divisor until they get to or close to the dividend.</td><td>192 ÷ 12 = 16
1 x 12 = 12
2 x 12 = 24
4 x 12 = 48
8 x 12 = 96
16 x 12 = 192
So, 192 ÷ 12 = 16</td></tr>
<tr><td>Partition or fair-share model
Students think of the number as the value of its place value, not as the independent digits.</td><td>192 ÷ 12 = 10 + 6 = 16
192 = 1 hundred, 9 tens, 2 ones
1 hundred cannot be shared by 12, so trade 1 hundred for 10 tens
10 tens + 9 tens = 19 tens
19 tens ÷ 12 = 10r7 (7 tens left over)
70 ones + 2 ones = 72 ones
72 ÷ 12 = 6</td></tr>
<tr><td>Partial quotient
Students consider fewer, large chunks of known problems involving the divisor to chunk the dividend.</td><td>1 x 12 = 12 or 12 ÷ 12 = 1
2 x 12 = 24 or 24 ÷ 12 = 2
10 x 12 = 120 or 120 ÷ 12 = 10
12 x 12 = 120 + 24 = 144 or 144 ÷ 12 = 12
16 x 12 = 144 + 24 + 24 = 196 or 196 ÷ 12 = 16
So, 192 ÷ 12
= (144 ÷ 12) + (24 ÷ 12) + (24 ÷ 12)
= 12 + 2 + 2 = 16</td></tr>
<tr><td>Partial quotient
(Over or under)
Students look for ways to manipulate numbers so that the calculations are easy.</td><td>24 ÷ 12 = 2
240 ÷ 12 = 20
48 ÷ 12 = 4
192 = 240 – 48
So, 192 ÷ 12
= (240 ÷ 12) – (48 ÷ 12) = 16</td></tr>
<tr><td>Repeated subtraction
Students use the standard algorithm and explicit-trade method as they work through the problem.</td><td>192 ÷ 12 = 16
<table>
<tr><td></td><td>0</td><td>1</td><td>6</td></tr>
<tr><td>12</td><td>1</td><td>9</td><td>2</td></tr>
<tr><td>-</td><td>0</td><td></td><td></td></tr>
<tr><td></td><td>1</td><td>9</td><td></td></tr>
<tr><td>-</td><td>1</td><td>2</td><td></td></tr>
<tr><td></td><td></td><td>7</td><td>2</td></tr>
<tr><td></td><td>-</td><td>7</td><td>2</td></tr>
<tr><td></td><td></td><td></td><td>0</td></tr>
</table></td></tr>
</table>

ANNEX C – "I NOTICE, I WONDER" ™

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Beginning to Problem Solve with "I Notice, I Wonder"™

I. What are some ways to get students started?

Activity 1: Basic "I Notice, I Wonder" Brainstorm

The obstacles: Students don't know how to begin solving word problems. They don't trust or make use of their own thinking. They freeze up or do any calculation that pops into their head, without thinking, "does this make sense?" They don't have ways to check their work or test their assumptions. They miss key information in the problem. They don't understand the "story" of the problem.

The solution: Create an safe environment where students focus on sharing their thoughts without any pressure to answer or solve a problem.

Display a problem scenario or complete problem at the front of the room. If reading level is a concern, read the scenario to students or have a volunteer read it.

Ask students, "What do you notice?"

Pause to let as many students as possible raise their hands. Call on students and record their noticings at the front of the room.

As you record students' thoughts, thank or acknowledge each student equally. Record all *student suggestions. Avoid praising, restating, clarifying, or asking questions.*

Ask students, "What are you wondering?"

Pause to let as many students as possible raise their hands. Call on students and record their wonderings at the front of the room.

Ask students, "Is there anything up here that you are wondering about? Anything you need clarified?" If you or the students have questions about any items, ask the students who shared them to clarify them further.

Activity 2: Forget the Question – Access for All Students

The obstacle: Sometimes when we put a problem on the board, students notice the question and go into one of two modes:

I don't understand, I'll never get this.

I know exactly what to do, let me work as quickly as I can.

This can make it difficult to facilitate a whole-group brainstorm. The first student doesn't participate and doesn't connect to his own thinking, losing out on the power of noticing and wondering. The second student doesn't participate and narrows in too quickly on her own thinking, losing out on the opportunity to surface more interesting (and more challenging) mathematical questions and ideas.

The solution: Use the basic "I Notice, I Wonder" Brainstorm, but include only the mathematical scenario. Leave out the question, and even some key information for solving the problem. Only after all students have participated and understand the scenario thoroughly do you reveal the question. Or, ask students, "If this story were the beginning of a math problem, what could the math problem be?" Then solve a problem the *students* came up with.

Leaving off the question increases participation from struggling students because there's no right answer and no wrong noticings and wonderings. It keeps speedy students engaged in creative brainstorming rather than closed-ended problem solving. And having a question to solve that students generated increases all students' understanding of the task and their engagement.

Activity 3: Think/Pair/Share – Increasing Engagement and Accountability

The obstacle: Some students are shy or hesitant to participate in a brainstorming session.

The solution: Hold all students accountable by giving each a recording sheet.

Students spend a minute (or more depending on their stamina) writing their noticings and wonderings on the recording sheet.

Students work with the person next to them to compare their lists and see if they can add two more things.

Each pair chooses one item to share with the whole group.

Quickly go around the room hearing each pair's items. Students should add noticings and wonderings they didn't come up with to their own sheets.

Finally ask, "Did anyone have any other noticings or wonderings they wanted to share?" and collect those.

In this fashion, each student is accountable for noticing and wondering before hearing from others, and students who are thoughtful and move slowly get a chance to organize their thoughts before sharing.

II. We Noticed, We Wondered, Now What?

Noticing and wondering is a tool to help students:

Understand the story, the quantities, and the relationships in the problem.
Understand what the problem is asking and what the answer will look like.
Have some ideas to begin to solve the problem.

This means that at the end of a noticing and wondering sessions, students should be able to:

Tell the story of the problem in their own words.
Give a reasonable estimate or high and low boundaries for the answer.
Work independently on carrying out steps or generating more data toward solving the problem.

If students are not ready to do those things, we recommend any of the following activities:

PoW IQ: Describe the *Information* and *Question*. Say what you are being asked to find, and estimate an answer. Give a high and low boundary for the answer, say whether it could be negative, fractional, zero, etc. Tell the key information given in the problem that you think you will use.

Act it Out: Have a group of students act out the problem while the audience looks at their list of noticing and wondering. The audience should be prepared to share new noticings and wonderings, as well as tell if the group missed or changed any noticings.

Draw a Picture: Have each student draw a sketch that they think shows what happens in the problems. They should sketch first and then label their picture. Students can then use their sketches to say the problem in their own words to a partner or small group.

III. Are We Done Noticing and Wondering Yet?

Noticings and wonderings are great tools for checking your work at the end of the problem. Students don't have to ask, "Am I correct?" They can look at their noticing, wondering, and estimates to make sure they were accountable to all the information in the problem.

And noticing and wondering is a skill students can get better at. That's why it's important to look back over your noticings and wonderings and ask, "Are we getting better?" After solving a problem, ask:

Which noticings and wonderings were really important to us?
Were there noticings and wonderings we didn't really use?
How do we come up with noticings and wonderings that are mathematical? What makes them mathematical?
Did we get stuck because we'd missed something? Why did we miss it? What could we do differently next time?

After noticing and wondering several times, ask:

Are there types of noticings and wonderings that are important? That we often miss?
Are we generating more noticings and wonderings each time? Are they getting more useful?
How do we go from noticings and wonderings to solution paths?

National Council of Teachers of Mathematics (NCTM). (n.d.). *Beginning to Problem Solve with "I Notice, I Wonder."* Nctm.org. Retrieved November 25, 2022, from https://www.nctm.org/Classroom-Resources/Problems-of-the-Week/I-Notice-I-Wonder/

ANNEX D – UNIT PLAN – PROBLEM-BASED LEARNING USING THE THREE-ACT TASK

I. Heading	Peng Seng, Yeo **Problem-based lesson** Mathematics, 4th grade 40 minutes
II. Rational and background	Essential understanding – Many real-world problems can be represented with a mathematical model, but that model may not represent a real-world situation exactly. Students use the 3-Act Math task to practice mathematical modelling. They: • Identify an important problem, • Identify the important information, • Develop a model that represents that situation, • Use the model to propose a solution, and • Test the appropriateness of that model. In this lesson, students draw on their conceptual understanding of addition and multiplication. They use representations and tools such as • Arrays, • Area models, and • Two- and three-dimensional shapes.
III. Lesson objective(s)	Students will be able to multiply a whole number of up to four digits by a one-digit whole number to solve problems.
IV. List of materials/resources	3-Act Math Recording Sheet
V. Procedures	[Before – Getting Ready] Act 1: The Hook https://www.youtube.com/watch?v=6l-mwNO8g2k The video shows a video of a group of people coving the walls of a room with square stickers. Take advantage of the student's initial reactions to watching the video. Ask, what do you notice about the video? What do you wonder? Encourage students to share their questions in a class discussion. Record their questions and store them later. Listen for interesting mathematical and non-mathematical questions. Ask, which question do you find most interesting? Which questions could we use mathematics to answer? Estimation **Main question: How many stickers do you need to cover the room's walls?**

	Point out that an estimation is only an estimate for the number of stickers needed to cover the box. Do not give students the time to make any calculations. Teacher to survey the class for a range of estimations. Point out that, without any information, expect a wide range of estimations. Ask, why do you think your estimation answers the main question? Who has a similar or different estimation? How many of you agree with the estimation? Make sure students it is important to think about low or high estimation to the main questions. What number is too small or too many to be the number of stickers?
	[During – Students work] Act 2: The Model Ask, what information do you need to answer the main question? The teacher provides information about what the students ask for. After discussing what information would be useful, ask, how could you get that information? How would you use it once you have it? Use images of the room to reveal each piece of information. Record information as students identify it and keep the information where students can refer to it. Have students discuss whether this information matches their expectations. There are 4 walls in the room. Two sides of the room are covered in a 123-by-7 array of stickers. The other two sides of the room are covered in a 19-by-7 array of stickers. Have students share their solution methods with the class. Ask, how did you use the slides of the room to solve the problem? How did you use the size of the stickers to solve the problem?
	[After – Class discussion] Act 3 – The Solution Reveal to the class that 1950 square stickers are needed to cover the 4 walls of the room. Ask, who got close to the answer? What did you or did not get close to the answer? What do you think you should do to get closer to the answer? Why are some estimations closer to the answer than others? To acknowledge that students have important ideas, use the remaining class time to return to students' list of questions and answer as many as possible.
	[Extension]

	If the room walls were twice as long, how many square stickers would you need to cover the four walls completely?
VI. Evaluation	At the end of the lesson, ask pupils to complete Daily Review 3-8 in their textbook.

ANNEX E – 3-Act Math Recording Sheet

Name: ________________________________ Date: ____________________

What did you notice?
What do you wonder?
Main question:
Make an estimate:
What information do you need?
Extension: Show your thinking
Create a similar word problem that matches the task: